MATH 5061 Problem Set 6[1]
Due date: May 1, 2020

Problems: (Please either type up your assignment or scan a copy of your written assignment into ONE PDF file and send it to me by email on/before the due date. Please remember to write down your name and SID. Questions marked with a † is optional.)

1. Prove that on any oriented closed (i.e. compact without boundary) Riemannian manifold (M^m, g), the Hodge Laplacian Δ commutes with d, δ and $*$ on $\Omega^p(M)$ for any p.

2. Let (Σ^2, g) be a closed oriented Riemannian surface with the induced complex structure J from g. Prove that for any vector field V on Σ, there exists a decomposition
$$V = \nabla f + J(\nabla g) + V_H$$
where $f, g \in C^\infty(\Sigma)$, and that locally $V_H = \nabla h$ for some (locally defined) harmonic function h.

3. Prove that for any closed p-form α_0 on a closed oriented Riemannian manifold (M^m, g), the harmonic representative α_H is the unique minimizer within its de Rham cohomology class $[\alpha_0] \in H^p_{dR}(M)$ for the Dirichlet energy:
$$E(\alpha) := \|\alpha\|^2_{L^2} = \int_M \langle \alpha, \alpha \rangle \, dV_g.$$

4. Let (M^m, g) be a closed Riemannian manifold. Let $\Gamma(S^2 M)$ denote the space of symmetric $(0,2)$-tensors on M. Consider the differential operator $S : \Omega^1(M) \to \Gamma(S^2 M)$ defined by
$$S(\alpha) := \mathcal{L}_{\alpha^\sharp} g$$
where $\alpha^\sharp$ is the vector field dual to α.
   (a) Prove that $\langle S\alpha, h \rangle_{L^2} = 2\langle \alpha, \delta h \rangle_{L^2}$ for any $\alpha \in \Omega^1(M)$, $h \in \Gamma(S^2 M)$. Here, $\delta : \Gamma(S^2 M) \to \Omega^1(M)$ is the divergence of a symmetric $(0,2)$-tensor.
   (b) Denote $D : \Omega^1(M) \to \Gamma(T^0_2 M)$ be the connection as defined in Q.2 of Problem Set 3, and D^* be its adjoint with respect to the L^2 inner product. Prove that for any $\alpha \in \Omega^1(M)$,
$$\delta S(\alpha) = D^* D\alpha + d\delta\alpha - \mathrm{Ric}(\alpha^\sharp, \cdot).$$
   (c) Use (b) to prove that the isometry group $\mathrm{Isom}(M)$ is finite if (M, g) has strictly negative Ricci curvature.

5. (†) Let (M^m, g) be a compact Riemannian manifold with smooth boundary. Suppose
   (i) M has non-negative Ricci curvature, i.e. $\mathrm{Ric} \geq 0$;
   (ii) The boundary ∂M has positive mean curvature (w.r.t. the outward unit normal ν), i.e. for any orthonormal basis $e_1, \cdots, e_{m-1}$ at ∂M,
$$H = \sum_{i=1}^{m-1} \langle D_{e_i} \nu, e_i \rangle > 0.$$
   Prove the inequality
$$\int_{\partial M} \frac{1}{H} \, dA \geq \frac{m}{m-1} \mathrm{Vol(M)}$$
   where dA is the $(m-1)$-dimensional volume form on ∂M and $\mathrm{Vol}(M)$ is the volume of (M^m, g). Show that equality is attained if and only if (M^m, g) is isometric to an m-dimensional Euclidean ball.

[1]Last revised on April 27, 2020